

# Remarks on the Geometry of Coordinate Projections in $\mathbb{R}^n$

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## Abstract

We study geometric properties of coordinate projections. Among other results, we show that if a body  $K \subset \mathbb{R}^n$  has an “almost extremal” volume ratio, then it has a projection of proportional dimension which is close to the cube. We also establish a sharp estimate on the shattering dimension of the convex hull of a class of functions in terms of the shattering dimension of the class itself.

## 1 Introduction

In this article we present several results on coordinate projections. The majority of this article is devoted to new applications of the entropy inequality established in [MV], which, roughly speaking, states that if a set of functions has a large entropy in  $L_2$ , it must have a coordinate projection which contains a large cube.

**Definition 1.1** *We say that a subset  $\sigma$  of  $\Omega$  is  **$t$ -shattered** by a class of real-valued functions  $F$  if there exists a level function  $h$  on  $\sigma$  such that, given any subset  $\sigma'$  of  $\sigma$ , one can find a function  $f \in F$  with  $f(x) \leq h(x) - t$  if  $x \in \sigma'$  and  $f(x) \geq h(x) + t$  if  $x \in \sigma \setminus \sigma'$ .*

*The **shattering dimension** of  $A$ , denoted by  $\text{vc}(F, \Omega, t)$  after Vapnik and Chervonenkis, is the maximal cardinality of a subset of  $\Omega$  which is  $t$ -shattered by  $F$ . In cases where the underlying space is clear we denote the shattering dimension by  $\text{vc}(F, t)$ .*

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**Theorem 1.2** *Let  $A$  be a class of functions bounded by 1, defined on a set  $\Omega$ . Then for every probability measure  $\mu$  on  $\Omega$ ,*

$$N(F, t, L_2(\mu)) \leq \left(\frac{2}{t}\right)^{K \cdot \text{vc}(F, ct)}, \quad 0 < t < 1, \quad (1.1)$$

where  $K$  and  $c$  are positive absolute constants.

Every  $F \subset \mathbb{R}^n$  can be identified with a class of functions on  $\{1, \dots, n\}$  in the natural way:  $v(i) = v_i$  for  $v \in F$ . If we take  $\mu$  to be the probability counting measure on  $\{1, \dots, n\}$  then (1.1) states that for any  $0 < t < 1$ ,

$$N(F, t\sqrt{n}B_2^n) \leq \left(\frac{2}{t}\right)^{K \cdot \text{vc}(F, ct)}.$$

Note that if  $F$  happens to be convex and symmetric with respect to the origin, then  $\text{vc}(F, t)$  is the maximal cardinality of a subset  $\sigma$  of  $\{1, \dots, n\}$  such that  $P_\sigma(F) \supset [-t, t]^\sigma$ .

We apply this result to study convex bodies whose volume ratio is almost maximal. Recall that the volume ratio, introduced by Szarek and Tomczak-Jaegermann [S, ST], is defined as  $\text{vr}(D) = (|D|/|\mathcal{E}|)^{1/n}$ , where  $\mathcal{E}$  is the ellipsoid of maximal volume contained in  $D$ .

The minimal volume ratio of a symmetric convex body in  $\mathbb{R}^n$  is 1 and is attained by the Euclidean ball; the maximal is of the order of  $\sqrt{n}$  and is attained by the cube  $B_\infty^n$  [B]. This pair of extremal bodies is unique up to a linear transformation. Indeed, the uniqueness of the minimizer is immediate, while the fact that cube is the unique maximizer was established in [Ba].

The isomorphic version of this fact – describing the bodies whose volume ratio is of order either 1 or  $\sqrt{n}$  is of particular interest. The question is whether such bodies inherit any structure from the Euclidean ball or, respectively, from the cube.

If the volume ratio of a body  $D$  in  $\mathbb{R}^n$  is bounded by a constant, then by the Volume Ratio Theorem [ST],  $D$  has a section of dimension proportional to  $n$ , which is well isomorphic to the Euclidean ball.

On the other hand, if  $\text{vr}(D)$  is of order of  $\sqrt{n}$ , then by [R] and [V],  $D$  has a section of dimension proportional to  $\sqrt{n}$ , which is well isomorphic to the cube, and the order of  $\sqrt{n}$  in the dimension can not be improved (the dual of Gluskin's polytope is such an example – see section 2). However, we show in section 2 that there exists a *projection* of  $D$  of dimension proportional to  $n$ , which is well isomorphic to the cube.

Two other applications we present are based on the following corollary of Theorem 1.2.

**Theorem 1.3** *[MV] Let  $F$  be a class of functions bounded by 1, defined on a finite set  $I$  of cardinality  $n$ . Then the gaussian process indexed by  $F$ ,  $X_f = \sum_{i=1}^n g_i f(i)$  satisfies*

$$E = \mathbb{E} \sup_{f \in F} X_f \leq K \sqrt{n} \int_{cE/n}^1 \sqrt{\text{vc}(F, t) \cdot \log(2/t)} dt,$$

where  $K$  and  $c$  are absolute constants.

One application we present is a comparison of the average  $\mathbb{E} \|\sum_{i=1}^n \varepsilon_i x_i\|$  to the minimum over all choices of signs,  $\min \|\sum_{i=1}^n \pm x_i\|$ . As a consequence, we compare the type 2 and the infratype 2 constants of a Banach space.

Then, we establish a sharp estimate on the shattering dimension of a convex hull of a class of functions, based on the shattering dimension of the class itself. Namely, we show that for every  $\varepsilon > 0$ ,

$$\text{vc}(\text{conv}(F), \varepsilon) \leq (C/\varepsilon)^2 \cdot \text{vc}(F, c\varepsilon),$$

where  $c$  and  $C$  are absolute constants.

The final question we address is when a random coordinate projection an “almost isometry”. Let  $(\Omega, \mu)$  be a probability space and  $f \in L_2(\mu)$ , and for simplicity, assume that  $\Omega = \{1, \dots, n\}$  and that  $\mu$  is the uniform probability measure on  $\Omega$ . For every  $\varepsilon > 0$ , our aim is to find “many” sets  $\sigma \subset \{1, \dots, n\}$  of small cardinality such that the natural coordinate projection  $P_\sigma f$  satisfies

$$(1 - \varepsilon) \|f\|_{L_2^n} \leq \|P_\sigma f\|_{L_2^\sigma} \leq (1 + \varepsilon) \|f\|_{L_2^n}, \quad (1.2)$$

where  $L_2^k$  is the  $L_2$  space defined on  $\{1, \dots, k\}$  with respect to the uniform probability measure.

By a standard concentration argument, if  $\|f\|_\infty \leq 1$ , then with high probability a random coordinate projection of dimension  $C/\varepsilon^2$  is an almost isometry in the sense of (1.2). We will show that the uniform boundedness of  $f$  can be relaxed; it suffices to assume that  $\|f\|_{\psi_2} \leq 1$ , where  $\|\cdot\|_{\psi_2}$  is the Orlicz norm generated by the function  $e^{t^2} - 1$ . In this case, a random coordinate projection of dimension  $C/\varepsilon^2$  will be an almost isometry as

in (1.2) with high probability. Although this result is relatively easy, we decided to present it because it gives hope that the conditions in stronger concentration inequalities (e.g. Talagrand’s concentration inequality for empirical processes [T 94, L]) can also be relaxed. As an application, we obtain a coordinate version of the Johnson-Lindenstrauss “Flattening” Lemma [JL].

Finally, we turn to some notational conventions. Throughout, all absolute constants are denoted by  $c$ ,  $C$ ,  $k$  and  $K$ . Their values may change from line to line or even within the same line. We denote  $a \sim b$  if there are absolute constants  $c$  and  $C$  such that  $cb \leq a \leq Cb$ .

**ACKNOWLEDGEMENTS:** The first author acknowledges partial support by an Australian Research Council Discovery Grant. The second author thanks Nicole Tomczak-Jaegermann for her constant support. A part of this work was done when the second author was visiting the Research School of Information Sciences and Engineering at The Australian National University, which he thanks for its hospitality. He also acknowledges a support from the Pacific Institute of Mathematical Sciences and is grateful to the Department of Mathematical Sciences of the University of Alberta for its hospitality.

## 2 Extremal volume ratios

The volume ratio of a convex body  $D$  in  $\mathbb{R}^n$  is defined as

$$\text{vr}(D) = \inf \left( \frac{|D|}{|\mathcal{E}|} \right)^{1/n},$$

where  $|\cdot|$  denotes the volume in  $\mathbb{R}^n$ , and the infimum is over all ellipsoids  $\mathcal{E}$  contained in  $D$ . This important invariant was introduced by Szarek and Tomczak-Jaegermann (see [S], [ST] or [P]).

The bodies with extremal volume ratios are the Euclidean ball and the cube – and these are the only extreme bodies up to a linear transformation (for the uniqueness of the cube, see [Ba]). One can show that for every convex symmetric body in  $\mathbb{R}^n$ ,

$$1 = \text{vr}(B_2^n) \leq \text{vr}(D) \leq \text{vr}(B_\infty^n), \quad (2.1)$$

(see [B]), while direct computation shows that  $\text{vr}(B_\infty^n) \leq C\sqrt{n}$  and the best value of the constant is  $C = 2/\sqrt{\pi e}$ .

Often, one encounters bodies whose volume ratio is *almost* extremal, i.e. close to one of sides of (2.1). The problem is whether such a body inherits properties of the extremal bodies, the Euclidean ball or the cube.

If  $\text{vr}(D) \leq A$ , then by the Volume Ratio Theorem [ST],  $D$  has a section of dimension  $k = n/2$  which is  $cA^2$ -isomorphic to the Euclidean ball  $B_2^k$ , and this result is asymptotically sharp.

On the opposite side of the scale, if  $\text{vr}(D) \geq A^{-1}\sqrt{n}$ ,  $D$  has a section of  $D$  of dimension  $k = c(A)\sqrt{n}$  which is  $C(A)\log n$ -isomorphic to the cube  $B_\infty^k$  [R, V]. It is not known whether the logarithmic term can be eliminated, but the order of  $\sqrt{n}$  in the dimension is optimal, as was noticed in [GTT]. Indeed, by an argument of Figiel and Johnson (see [FJ], cor. 3.2), a random subspace  $E \subset \ell_\infty^n$  (and thus a dual of Gluskin's space) of dimension at least  $n/2$  satisfies that for any  $F \subset E$ ,  $gl(F) \geq c\dim(F)/\sqrt{n}$ , where  $gl(F)$  is the Gordon-Lewis constant of  $F$ , and  $c$  is a suitable absolute constant. By [GL],  $gl(F) \leq \text{unc}(F)$ , where  $\text{unc}(F)$  is the least unconditionality constant of a basis of  $F$ . Since  $\text{unc}(\ell_\infty^k) = 1$  then

$$d(F, \ell_\infty^k) \geq \text{unc}(F) \geq \frac{c\dim(F)}{\sqrt{n}},$$

and thus, if  $F$  is 2-isomorphic to  $\ell_\infty^k$  then  $\dim(F) \leq c'\sqrt{n}$ .

Our next result shows that  $D$  has a *projection* of dimension proportional to  $n$  which is  $cA$ -isomorphic to the cube  $B_\infty^k$ .

**Theorem 2.1** *There are absolute constants  $C$  and  $c$  for which the following holds. If  $D$  is a convex symmetric body in  $\mathbb{R}^n$  for which  $\text{vr}(D) \geq A^{-1}\sqrt{n}$ , then there exists a projection  $P$  of rank  $k \geq cn/\log A$  such that*

$$d(PK, B_\infty^k) \leq CA.$$

To prove the Theorem, recall the notion of the *cubic ratio* [B]. For every ball  $D \subset \mathbb{R}^n$  one defines

$$\text{cr}(D) = \inf \left( \frac{|B_\infty^n|}{|TD|} \right)^{\frac{1}{n}},$$

where the infimum is over all linear invertible operators  $T$  on  $\mathbb{R}^n$  such that  $TD \subset B_\infty^n$ .

**Lemma 2.2** [B] *There are absolute constants  $c$  and  $C$  such that for every integer  $n$  and every convex symmetric body  $D \subset \mathbb{R}^n$ ,*

$$c\sqrt{n} \leq \text{vr}(D) \cdot \text{cr}(D) \leq C\sqrt{n}.$$

**Proof of Theorem 2.1.** Clearly, we can assume  $n$  to be larger than a suitable absolute constant  $N$ , which ensures that for every  $D \subset \mathbb{R}^n$ ,  $\text{vr}(D) \leq 0.8\sqrt{n}$ . Since  $\text{vr}(D) \geq A^{-1}\sqrt{n}$ , then by Lemma 2.2,  $\text{cr}(D) \leq CA$ . Hence, there is some  $T \in GL_n$  such that

$$TD \subset B_\infty^n \quad \text{and} \quad |TD|^{\frac{1}{n}} \geq c/A.$$

Recall that  $c_1^n \leq |\sqrt{n}B_2^n| \leq c_2^n$  for some absolute constants  $c_1, c_2$ , and thus there exists an absolute constant  $c_3$  such that

$$2^n = \frac{|TD|}{|c_3 A^{-1}(\sqrt{n}B_2^n)|}.$$

By a standard volumetric argument, the right-hand side is bounded by

$$N(TD, c_3 A^{-1} \sqrt{n} B_2^n),$$

and by Theorem 1.2 there are absolute constants  $K$  and  $c$  for which

$$n \leq \log N(TD, c_3 A^{-1} \sqrt{n} B_2^n) \leq K \cdot \text{vc}(TD, cA^{-1}) \log(CA).$$

Hence, there is a set  $\sigma \subset \{1, \dots, n\}$ , such that  $|\sigma| \geq n/K \log(CA)$  and

$$c_1 A^{-1} B_\infty^\sigma \subset P_\sigma(TD) \subset B_\infty^\sigma.$$

It only remains to note that  $\log(CA) \leq C' \log A$ , because  $A \geq 5/4$ . ■

**Remark.** Since the volume ratio is always greater than 1, then  $A \geq n^{-1/2}$ . Therefore, the dimension of the cubic projection in Theorem 2.1 is always bounded below by  $cn/\log n$ .

In a very similar way, one can prove the following

**Theorem 2.3** *There are absolute constants  $C$  and  $c$  for which the following holds. Let  $D$  be a convex symmetric body in  $\mathbb{R}^n$  for which  $\text{vr}(D) \geq A^{-1}\sqrt{n}$ . Then there exists a projection  $P$  of rank  $k \geq cn$  such that*

$$d(PK, B_\infty^k) \leq CA^2.$$

### 3 Type and Infratype

In this section we improve a result of M. Talagrand [T 92] which compares the average over the  $\pm$  signs to the minimum over the  $\pm$  signs of  $\|\sum_{i=1}^n \pm x_i\|$ .

Recall that a Banach space  $X$  has a (gaussian) type  $p$  if there exists some  $M > 0$  such that for all  $n$  and all sequences of vectors  $(x_i)_{i \leq n}$ ,

$$\mathbb{E} \left\| \sum_{i=1}^n g_i x_i \right\| \leq M \left( \sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}}. \quad (3.1)$$

The best possible constant  $M$  in this inequality is denoted by  $T_p(X)$ . We say that  $X$  has infratype  $p$  if there exists some  $M > 0$  such that for all  $n$  and all sequences of vectors  $(x_i)_{i \leq n}$ ,

$$\min_{\eta_i = \pm 1} \left\| \sum_{i=1}^n \eta_i x_i \right\| \leq M \left( \sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}}. \quad (3.2)$$

The best possible constant  $M$  in this inequality is denoted by  $I_p(X)$ .

In [T 92] it was shown that if  $1 < p < 2$  then  $T_p(X) \leq C_p I_p(X)^2$ , where  $C_p$  is a constant which depends only on  $p$ . It is not known whether the square can be removed. Regarding the case  $p = 2$ , M. Talagrand recently constructed a symmetric sequence space which has infratype 2 but not type 2 [T 03]. Hence one can not obtain dimension free estimates on  $T_2(X)$  in terms of  $I_2(X)$ . Our main result in this section is that if  $\dim(X) = n$  then  $T_2(X) \leq C I_2(X) \cdot \log^{3/2} n$ .

We begin with the following fact that allows one to compare Rademacher and Gaussian averages.

**Lemma 3.1** *There is an absolute constant  $C$  for which the following holds. Let  $x_1, \dots, x_n$  be vectors in the unit ball of a Banach space and let  $0 < M \leq \sqrt{n}$ . If  $0 < \lambda < \log^{-3}(n/M^2)$  and*

$$\min_{\eta_i = \pm 1} \left\| \sum_{i \in \sigma} \eta_i x_i \right\| \leq M |\sigma|^{\frac{1}{2}}$$

*for all  $\sigma \subset \{1, \dots, n\}$  with  $|\sigma| \leq \lambda n$ , then,*

$$\mathbb{E} \left\| \sum_{i=1}^n g_i x_i \right\| \leq C M (n/\lambda)^{1/2}.$$

In the proof of Lemma 3.1 we require the following observation from [MS], that if  $\{x_1, \dots, x_n\} \subset X$  is  $\varepsilon$ -shattered by  $B_{X^*}$ , then for any  $(a_i)_{i=1}^n \in \mathbb{R}^n$ ,

$$\varepsilon \sum_{i=1}^n |a_i| \leq \left\| \sum_{i=1}^n a_i x_i \right\|. \quad (3.3)$$

**Proof of Lemma 3.1.** Clearly we can assume that the given Banach space is  $X = (\mathbb{R}^n, \|\cdot\|)$  and that  $(x_i)_{i \leq n}$  are the unit coordinate vectors in  $\mathbb{R}^n$ . Set  $B = B_{X^*}$  and by the hypothesis of the lemma and (3.3),  $\text{vc}(B, Mv^{-1/2}) \leq v$  if  $0 \leq v \leq \lambda n$ . Hence, for any  $M(\lambda n)^{-1/2} \leq t \leq 1$ ,

$$\text{vc}(B, t) \leq (M/t)^2. \quad (3.4)$$

Set

$$E = \mathbb{E} \left\| \sum_{i=1}^n g_i x_i \right\|_X = \mathbb{E} \sup_{b \in B} \sum_{i=1}^n g_i b(i).$$

By Theorem 1.3, there are absolute constants  $C$  and  $c$  such that

$$E \leq K \sqrt{n} \int_{cE/n}^1 \sqrt{\text{vc}(B, t) \cdot \log(2/t)} dt.$$

If  $cE/n \leq M(\lambda n)^{-1/2}$ , the lemma trivially follows. Otherwise, if the converse inequality holds, then by (3.4) and since  $\lambda < 1$ ,

$$E \leq K \sqrt{n} \int_{cE/n}^1 (M/t) \sqrt{\log(2/t)} dt \leq K \sqrt{n} M \cdot \log^{\frac{3}{2}}(n/M^2),$$

and by the assumption on  $\lambda$ ,

$$E \leq K \sqrt{n} M \cdot \log^{\frac{3}{2}}(n/M^2) \leq K \sqrt{n} M / \sqrt{\lambda},$$

as claimed. ■

Using Lemma 3.1, one can compare the type and infratype 2 of a Banach space  $X$ .

Let  $T_2^{(n)}(X)$  and  $I_2^{(n)}(X)$  denote the best possible constants  $M$  in (3.1) and (3.2) respectively (with  $p = 2$ ). So,  $T_2^{(n)}(X)$  and  $I_2^{(n)}(X)$  measure the type/infratype 2 computed on  $n$  vectors. Clearly,  $I_2(X) \leq T_2(X)$  and  $I_2^{(n)}(X) \leq T_2^{(n)}(X) \leq \sqrt{n}$ .

**Theorem 3.2** *Let  $X$  be an  $n$ -dimensional Banach space. Then, for every number  $0 < \lambda < \log^{-3}(n/I_2(X)^2)$ ,*

$$T_2(X) \leq C \cdot I_2^{(\lambda n)}(X) / \sqrt{\lambda}.$$



In particular,

$$T_2(X) \leq I_2(X) \cdot C \log^{\frac{3}{2}} \left( \frac{n}{I_2(X)^2} \right) \leq I_2(X) \cdot C \log^{\frac{3}{2}} n.$$

**Proof.** By [TJ] and [BKT] Theorem 3.1, the Gaussian type 2 can be computed on  $n$  vectors of norm one. Precisely, this means that  $T_2(X)$  is the smallest possible constant  $M'$  for which the inequality

$$\mathbb{E} \left\| \sum_{i=1}^n g_i x_i \right\| \leq M' n^{1/2}$$

holds for all vectors  $x_1, \dots, x_n$  of norm one. Now, the assertion follows from Lemma 3.1.  $\blacksquare$

## 4 The shattering dimension of convex hulls

In this section we present a sharp estimate which compares the shattering dimensions of a class and of its convex hull. To that end, we connect the shattering dimension to the growth rate of the expectation of the supremum of the gaussian process  $\{X_a, a \in P_\sigma F\}$  as a function of  $|\sigma|$ .

**Definition 4.1** *Let  $F$  be a class of functions bounded by 1 and set*

$$\ell_n(F) = \sup_{(x_1, \dots, x_n) \in \Omega^n} \mathbb{E}_g \sup_{f \in F} \left| \sum_{i=1}^n g_i f(x_i) \right|,$$

*where  $g_1, \dots, g_n$  are independent, standard gaussian random variables.*

Hence,  $\ell_n(F)$  is the largest gaussian average associated with a coordinate projection of  $F$  on  $n$  points. Since  $F \subset B(L_\infty(\Omega))$  then for every  $\sigma = (x_1, \dots, x_n)$ ,

$$P_\sigma F = \left\{ (f(x_1), \dots, f(x_n)) : f \in F \right\} \subset B_\infty^n,$$

and the largest projection one might encounter is when  $P_\sigma F = B_\infty^n$ , in which case  $\ell(P_\sigma F) \sim n$ . We define a scale-sensitive parameter which measures for every  $\varepsilon > 0$  the largest cardinality of a projection which has a “large”  $\ell$ -norm:

$$t(F, \varepsilon) = \sup \{n : \ell_n(F) \geq \varepsilon n\}.$$

**Theorem 4.2** *There are absolute constants  $K$ ,  $c$  and  $c'$  such that for any  $F \subset B(L_\infty(\Omega))$  and every  $\varepsilon > 0$ ,*

$$\text{vc}(F, c'\varepsilon) \leq t(F, \varepsilon) \leq (K/\varepsilon^2) \cdot \text{vc}(F, c\varepsilon).$$

In the proof, we will use the following wording: the function  $f$  associated to a set  $\sigma'$  in the Definition 1.1 will be called the **function that shatters**  $\sigma'$ .

**Proof of Theorem 4.2.** Assume that  $\{x_1, \dots, x_n\}$  is  $\varepsilon$ -shattered by  $F$ . For every  $J \subset \{x_1, \dots, x_n\}$ , let  $f_J$  be the function shattering  $J$ , and for each  $(\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$  set  $I = \{x_i | \varepsilon_i = 1\}$ . By the triangle inequality and letting  $f = f_I$ ,  $f' = f_{I^c}$  in the second inequality below,

$$\begin{aligned} \sup_{f \in F} \left| \sum_{i=1}^n \varepsilon_i f(x_i) \right| &\geq \frac{1}{2} \sup_{f, f' \in F} \left| \sum_{i=1}^n \varepsilon_i (f(x_i) - f'(x_i)) \right| \\ &\geq \frac{1}{2} \left| \sum_{i=1}^n \varepsilon_i (f_I(x_i) - f_{I^c}(x_i)) \right| \geq n\varepsilon. \end{aligned}$$

Hence,

$$\sup_{f \in F} \left| \sum_{i=1}^n \varepsilon_i f(x_i) \right| \geq n\varepsilon,$$

and in particular this holds for the average. The first bound is evident because of the known connections between gaussian and Rademacher averages [TJ1], namely, that there is an absolute constant  $C$  such that for any class  $F$  and any set  $\sigma = (x_1, \dots, x_n)$ ,

$$\begin{aligned} \ell(P_\sigma F) &= \mathbb{E}_g \left\| \sum_{i=1}^n g_i e_i \right\|_{(P_\sigma F)^\circ} \geq C \cdot \mathbb{E}_\varepsilon \left\| \sum_{i=1}^n \varepsilon_i e_i \right\|_{(P_\sigma F)^\circ} \\ &= C \cdot \mathbb{E}_\varepsilon \sup_{f \in F} \left| \sum_{i=1}^n \varepsilon_i f(x_i) \right|. \end{aligned}$$

The reverse inequality follows from Theorem 1.3 in a similar way to the proof of Elton's Theorem in [MV]. If  $\ell(P_\sigma F) \geq \varepsilon n$ , then  $\mathbb{E} \sup_{f \in F} X_f \geq n\varepsilon$ , where  $X_f = \sum_{i=1}^n g_i f(x_i)$ . By Theorem 1.3,

$$n\varepsilon \leq \mathbb{E} \sup_{f \in F} X_f \leq K\sqrt{n} \int_{c\varepsilon}^1 \sqrt{\text{vc}(P_\sigma F, t) \cdot \log(2/t)} dt.$$

Set  $v(t) = \frac{c_0}{t \log^{1.1}(2/t)}$  where  $c_0 > 0$  is chosen so that  $\int_0^1 v(t) dt = 1$ . Hence, there is some  $c\varepsilon \leq t \leq 1$  such that  $K^2 \cdot \text{vc}(P_\sigma F, t) \geq \varepsilon^2 n \cdot v^2(t)/\log(2/t)$ , implying that

$$\text{vc}(F, c\varepsilon) \geq \text{vc}(P_\sigma F, c\varepsilon) \geq \text{vc}(P_\sigma F, t) \geq \frac{c'\varepsilon^2}{t^2 \log^{3.2}(2/t)} n \geq c'' n \varepsilon^2.$$

■

The previous result can be used to estimate the shattering dimension of a convex hull of a class.

**Corollary 4.3** *There are absolute constants  $K$  and  $c$  such that for any  $F \subset B(L_\infty(\Omega))$  and every  $\varepsilon > 0$ ,*

$$\text{vc}(\text{conv}(F), \varepsilon) \leq (K/\varepsilon)^2 \cdot \text{vc}(F, c\varepsilon).$$

**Proof.** Since the  $\ell$ -norm of a set and of its convex hull are the same, then for any  $\varepsilon > 0$ ,  $t(F, \varepsilon) = t(\text{conv}(F), \varepsilon)$ . By Theorem 4.2,

$$\text{vc}(\text{conv}(F), \varepsilon) \leq t(\text{conv}(F), \varepsilon) = t(F, \varepsilon) \leq (K/\varepsilon)^2 \cdot \text{vc}(F, c\varepsilon).$$

■

Next, we show that this estimate is sharp, in the sense that the exponent of  $1/\varepsilon^2$  can not be improved. To that end, we require some properties of the shattering dimension of classes of linear functionals mentioned before, which was investigated in [MS].

If  $X$  is a normed space then  $B_{X^*}$  can be viewed as a subset of  $L_\infty(B_X)$  in the natural way. It is not difficult to characterize the shattering dimension in this case.

**Lemma 4.4** *A set  $\{x_1, \dots, x_n\} \subset B_X$  is  $\varepsilon$ -shattered by  $B_{X^*}$  if and only if  $(x_i)_{i=1}^n$  are linearly independent and  $\varepsilon$ -dominate the  $\ell_1^n$  unit-vector basis; i.e.,*

$$\varepsilon \sum_{i=1}^n |a_i| \leq \left\| \sum_{i=1}^n a_i x_i \right\| \leq \sum_{i=1}^n |a_i|$$

for every  $a_1, \dots, a_n \in \mathbb{R}$ .

*In particular, if  $X$  is  $n$ -dimensional, and if the Banach-Mazur distance satisfies that  $d(X, \ell_1^n) \leq \alpha$ , then  $\text{vc}(B_{X^*}, B_X, 1/\alpha) = n$ .*

**Corollary 4.5** *There exists an absolute constant  $k$  for which the following holds. For every  $0 < \varepsilon < 1/2$  there is a class  $F \subset B(L_\infty(\Omega))$  such that*

$$\text{vc}(\text{conv}(F), \varepsilon) \geq \frac{k \cdot \text{vc}(F, \varepsilon)}{\varepsilon^2 \log(1/\varepsilon)}.$$

**Proof.** For every integer  $n$ , let  $\Omega_n = B_\infty^n$  and set  $F_n = \{e_1, \dots, e_n\}$ , that is, the standard unit vectors in  $\mathbb{R}^n$ , when considered as linear functionals on  $B_\infty^n$ . Since  $|F| = n$ , it follows that for every  $\varepsilon > 0$ ,  $\text{vc}(F_n, \Omega_n, \varepsilon) \leq \log_2 n$ . On the other hand,  $\text{conv}(F_n) = B_1^n$  when considered as functionals on  $B_\infty^n$ . By Lemma 4.4 applied to  $X = \ell_\infty^n$ , and since  $d(\ell_\infty^n, \ell_1^n) \leq K\sqrt{n}$  [TJ1], it is evident that there is a subset on cardinality  $n$  in  $B_\infty^n$  which is  $k/\sqrt{n}$ -shattered by  $B_1^n$ . Thus, for  $\varepsilon_n = k/\sqrt{n}$ ,

$$\text{vc}(\text{conv}(F_n), \Omega_n, \varepsilon_n) \geq \frac{k' \cdot \text{vc}(F_n, \Omega_n, \varepsilon_n)}{\varepsilon_n^2 \log(1/\varepsilon_n)},$$

from which the proof easily follows. ■

## 5 Almost isometric coordinate projections

Given real-valued function  $f$  on a probability space, its  $\psi_p$ -norm ( $p \geq 1$ ) is defined as the Orlicz norm corresponding to the function  $\exp(t^p) - 1$ . Precisely,  $\|f\|_{\psi_p}$  is the infimum of all numbers  $\lambda$  satisfying  $\mathbb{E} \exp(|f|^p/\lambda^p) \leq e$ . It is possible to compare the  $\psi_p$  with other  $\psi_q$  norms and the  $L_p$  norms. Indeed, one can show that if  $1 \leq p \leq q < \infty$ ,  $\|f\|_{\psi_p} \leq C_{p,q} \|f\|_{\psi_q}$ , and  $\|f\|_{L_p} \leq C_p \|f\|_{\psi_1}$  (see, for example, [VW]).

A function  $f$  is bounded in the  $\psi_2$  norm if and only if  $f$  has a subgaussian tail. Namely, if  $\|f\|_{\psi_2} \leq 1$  then by Chebychev's inequality  $\mathbb{P}\{|f| > t\} \leq e^{-t^2+1}$  for all  $t > 0$ . Conversely, if for some  $A \geq 1$  one has  $\mathbb{P}\{|f| > t\} \leq Ae^{-t^2}$  for all  $t > 1$ , then integrating by parts it follows that  $\mathbb{E} \exp(f/2)^2 \leq 1 + A/3 \leq 2^A$ , and by Jensen's inequality one can conclude that  $\|f\|_{\psi_2} \leq 2A$  (we did not attempt here to give the right dependence on  $A$ ).

Another simple but useful fact which follows from Jensen's inequality is that  $\|f\|_{\psi_2} \leq C \mathbb{E} \exp(f^2)$ , where  $C$  is an absolute constant.

We will focus on functions defined on a finite domain, which we identify with  $\{1, \dots, n\}$ , equipped with a uniform measure, where each atom carries a weight of  $1/n$ . We denote the  $\psi_2$  norm of a function  $f$  on this

probability space by  $\|f\|_{\psi_2^n}$ . Since  $f$  is defined on  $\{1, \dots, n\}$ , we sometimes identify  $f$  with the sequence of scalars  $(f(i))_{i=1}^n$ .

We shall use the following standard probabilistic model for random coordinate projections. Given  $0 < \delta \leq 1/2$ , let  $\delta_1, \dots, \delta_n$  be selectors, i.e. independent  $\{0, 1\}$ -valued random variables with mean  $\delta$ . Then  $\sigma = \{i \mid 1 \leq i \leq n, \delta_i = 1\}$  is a random subset of the interval  $\{1, \dots, n\}$  with average cardinality  $\delta n$ .

By Bernstein's inequality [VW], for every  $0 < \varepsilon < 1$ ,

$$\mathbb{P}\left\{(1 - \varepsilon)\|f\|_{L_2^n}^2 \leq \frac{1}{\delta n} \sum_{i=1}^n \delta_i |f(i)|^2 \leq (1 + \varepsilon)\|f\|_{L_2^n}^2\right\} \geq 1 - 2 \exp\left(-\frac{c\varepsilon^2 \delta n}{\|f\|_\infty}\right),$$

and by another application of Bernstein's inequality,

$$\mathbb{P}\left\{\frac{1}{\delta n} \sum_{i=1}^n |\delta_i - \delta| \geq \varepsilon\right\} \leq 2 \exp(-c\varepsilon^2 n \delta), \quad (5.1)$$

implying that if  $\|f\|_\infty \leq 1$ , then with probability at least  $4 \exp(-c\varepsilon^2 |\sigma|)$ ,

$$(1 - \varepsilon)\|f\|_{L_2^n} \leq \|P_\sigma f\|_{L_2^\sigma} \leq (1 + \varepsilon)\|f\|_{L_2^n}.$$

In this section we relax the assumption that  $f$  is bounded in the uniform norm, and assume that  $f$  is bounded in the  $\psi_2$  norm.

Roughly speaking, we show that for every  $1 \leq p < \infty$ , the set of vectors in  $S(L_p^n)$  which will be almost isometrically projected onto  $L_p^\sigma$  are those with a "small"  $\psi_p^n$  norm.

**Proposition 5.1** *Let  $(\delta_i)_{i=1}^n$  be independent  $\{0, 1\}$ -valued random variables with mean  $\delta > 0$ . Set  $a = (a_i)_{i=1}^n \in \mathbb{R}^n$  and put  $M = \|a\|_{\psi_1^n}$ . Then, for every positive number  $t < M/2$ ,*

$$\mathbb{P}\left\{\sum_{i=1}^n (\delta_i - \delta) a_i > t \delta n\right\} \leq \exp\left(-\frac{ct^2 \delta n}{M^2}\right),$$

where  $c$  is an absolute constant.

The proof starts with the following standard lemma.

**Lemma 5.2** *Let  $Z$  be a random variable and assume that for some  $b, \lambda > 0$ ,*

$$\mathbb{E} \exp(\lambda Z) \leq e^{b^2 \lambda^2}.$$

*Then*

$$\mathbb{P}\{Z > 2b^2 \lambda\} \leq e^{-b^2 \lambda^2}.$$

**Proof.** For  $t > 0$ ,

$$\begin{aligned}\mathbb{P}\{Z > t\} &= \mathbb{P}\{\exp(\lambda(Z - t)) > 1\} \leq \mathbb{E} \exp(\lambda(Z - t)) \\ &= e^{-\lambda t} \mathbb{E} \exp(\lambda Z) \leq e^{b^2 \lambda^2 - \lambda t}.\end{aligned}$$

Setting  $t = 2b^2 \lambda$  completes the proof.  $\blacksquare$

**Proof of Proposition 5.1.** By homogeneity, we can assume that  $M = 1$ , and we shall evaluate  $\mathbb{E} \exp(t \sum_{i=1}^n (\delta_i - \delta) a_i)$ . To that end, let  $\delta'_i$  be an independent copy of  $\delta_i$  and set  $\tilde{\delta}_i = \delta_i - \delta'_i$ . By Jensen's inequality,

$$\mathbb{E} \exp\left(t \sum_{i=1}^n (\delta_i - \delta) a_i\right) \leq \mathbb{E} \exp\left(t \sum_{i=1}^n (\delta_i - \delta'_i) a_i\right) = \prod_{i=1}^n \mathbb{E} \exp(t \tilde{\delta}_i a_i) = E.$$

Set  $\tilde{\delta} = \delta(1 - \delta)$  and note that  $\tilde{\delta}_i$  is 0 with probability  $1 - 2\tilde{\delta}$ , and 1 and  $-1$ , each with probability  $\tilde{\delta}$ . Therefore,

$$\mathbb{E} \exp(t \tilde{\delta}_i a_i) = (1 - 2\tilde{\delta}) + \tilde{\delta} e^{ta_i} + \tilde{\delta} e^{-ta_i} = 1 + 2\tilde{\delta}(\cosh(ta_i) - 1).$$

Since  $\cosh x \leq 1 + \frac{1}{2}x^2 e^{|x|}$  for all real  $x$ , then

$$E \leq \prod_{i=1}^n (1 + \tilde{\delta} t^2 a_i^2 e^{t|a_i|}) \leq \prod_{i=1}^n \exp(\tilde{\delta} t^2 a_i^2 e^{t|a_i|}) = \exp\left(\tilde{\delta} t^2 n \cdot \frac{1}{n} \sum_{i=1}^n a_i^2 e^{t|a_i|}\right).$$

The normalized sum is estimated by Cauchy-Schwartz and using the fact that  $2t \leq 1$ :

$$\begin{aligned}\frac{1}{n} \sum_{i=1}^n a_i^2 e^{t|a_i|} &\leq \left(\frac{1}{n} \sum_{i=1}^n |a_i|^4\right)^{\frac{1}{2}} \left(\frac{1}{n} \sum_{i=1}^n e^{2t|a_i|}\right)^{\frac{1}{2}} \\ &\leq \|a\|_{L_4^n}^2 \left(\frac{1}{n} \sum_{i=1}^n e^{|a_i|}\right)^{\frac{1}{2}} \\ &\leq 2\|a\|_{L_4^n}^2 \leq C,\end{aligned}$$

because  $c\|a\|_{L_4^n} \leq \|a\|_{\psi_1^n} \leq 1$ . Hence,

$$E \leq \exp(C \tilde{\delta} t^2 n) \leq \exp(C' \delta t^2 n).$$

We put this in a form convenient for applying Lemma 5.2:

$$\mathbb{E} \exp\left(t \delta n \cdot \frac{1}{\delta n} \sum_{i=1}^n (\delta_i - \delta) a_i\right) \leq \exp\left(\frac{C'}{\delta n} (t \delta n)^2\right)$$

and apply the lemma for  $\lambda = t\delta n$  and  $b^2 = \frac{C'}{\delta n}$ . It follows that for every  $t > 0$ ,

$$\mathbb{P}\left\{\frac{1}{\delta n} \sum_{i=1}^n (\delta_i - \delta) a_i > 2ct\right\} \leq \exp(-c\delta t^2 n),$$

which completes the proof.  $\blacksquare$

**Corollary 5.3** *Applying Proposition 5.1 to  $-a_i$  it is evident that*

$$\mathbb{P}\left\{\left|\sum_{i=1}^n (\delta_i - \delta) a_i\right| > t\delta n\right\} \leq 2 \exp\left(-\frac{ct^2 \delta n}{M^2}\right)$$

where  $M = \|a\|_{\psi_1^n}$  and  $0 < t < M/2$ .

An easy application of this corollary is the fact that the  $\psi_2^n$ -norm of points on the sphere determines the cardinality of an almost isometric projection.

**Corollary 5.4** *There is an absolute constant  $C$  for which the following holds. For every integer  $n$ , any  $f \in S(L_2^n)$  and every  $\varepsilon > 0$ , a random set  $\sigma \subset \{1, \dots, n\}$  of average cardinality  $(CM/\varepsilon)^2$  satisfies with probability at least  $1/2$  that*

$$1 - \varepsilon \leq \|P_\sigma f\|_{L_2^\sigma} \leq 1 + \varepsilon,$$

where  $M = \|f\|_{\psi_2^n}$ .

**Proof.** The proof follows immediately from Corollary 5.3, by taking  $a_i = f^2(i)$  and  $\delta n = (CM/\varepsilon)^2$ , and applying (5.1).  $\blacksquare$

Note that a similar result can be easily derived for any  $1 \leq p < \infty$ , simply by the fact that  $\|(a_i)\|_{\psi_p^n} = \|(a_i^p)\|_{\psi_1^n}$ .

Corollary 5.3 can be used to present a new insight to the well known Johnson-Lindenstrauss “Flattening” Lemma [JL], which states that every set  $\{x_1, \dots, x_n\} \subset \ell_2^n$  can be  $1 + \varepsilon$  isometrically embedded in  $\ell_2^m$ , where  $m \leq (C/\varepsilon)^2 \log n$ . One can formulate the Johnson-Lindenstrauss Lemma as follows:

**Theorem 5.5** *There is an absolute constant  $C$  for which the following holds. For every  $f_1, \dots, f_n \in S(L_2^n)$  and every  $\varepsilon > 0$  there is an orthogonal operator  $O$  and a set  $\sigma \subset \{1, \dots, n\}$  of cardinality at most  $(C/\varepsilon)^2 \log n$ , such that for all  $1 \leq i \leq n$ ,*

$$1 - \varepsilon \leq \|P_\sigma O f_i\|_{L_2^\sigma} \leq 1 + \varepsilon.$$

As Corollary 5.4 shows, an almost isometric coordinate projection of  $f$  is possible, as long as  $\|f\|_{\psi_2^n}$  is small; hence, the  $\psi_2^n$  norm defines a “good region” on the sphere for which a random coordinate projection will be an almost isometry. In a similar way, this can also be performed with many functions simultaneously:

**Corollary 5.6** *There is an absolute constant  $C$  for which the following holds. For every  $f_1, \dots, f_n \in S(L_2^n)$  and every  $\varepsilon > 0$  a random set  $\sigma \subset \{1, \dots, n\}$  of cardinality  $(CM/\varepsilon)^2 \log n$  satisfies that with probability at least  $1/2$ ,*

$$1 - \varepsilon \leq \|P_\sigma f_i\|_{L_2^\sigma} \leq 1 + \varepsilon, \quad 1 \leq i \leq n.$$

where  $M = \max_i \|f_i\|_{\psi_2^n}$ .

**Proof.** As in Corollary 5.4, but taking  $\delta n = (CM/\varepsilon)^2 \log n$ , we obtain then for every  $1 \leq i \leq n$

$$Pr\{1 - \varepsilon \leq \|P_\sigma f_i\|_{L_2^\sigma} \leq 1 + \varepsilon\} \geq 1 - \frac{1}{2n}.$$

Then

$$Pr\{\forall 1 \leq i \leq n, \quad 1 - \varepsilon \leq \|P_\sigma f_i\|_{L_2^\sigma} \leq 1 + \varepsilon\} \geq 1/2,$$

which completes the proof. ■

The connection to the Johnson-Lindenstrauss Lemma is easy: with high probability, a random orthogonal operator  $O$  will map any set of  $n$  vectors on the sphere to the “good region”, i.e. to the region where the  $\psi_2^n$  norm is bounded by an absolute constant.

**Lemma 5.7** *There is an absolute constant  $C$  such that for every integer  $n$  and any  $x \in S(L_2^n)$ ,*

$$Pr_{O_n}\{\|Ox\|_{\psi_2^n} \geq C\} < \frac{1}{2n},$$

where the probability measure is the Haar measure on the orthogonal group.

As a consequence, for every  $f_1, \dots, f_n \in S(L_2^n)$ ,

$$\max_i \|Of_i\|_{\psi_2^n} \leq C$$

with probability greater than  $1/2$ , and thus Theorem 5.5 is implied by Corollary 5.6.



**Proof.** Clearly, it suffices to show that there is an absolute constant  $C$  such that

$$Pr\{x \in S^{n-1} : \|x\|_{\psi_2^n} \geq \frac{C}{\sqrt{n}}\} \leq \frac{1}{2n}.$$

Consider the function  $g : S^{n-1} \rightarrow \mathbb{R}$  defined by  $g(x) = \|x\|_{\psi_2^n}$ . To estimate its Lipschitz constant, observe that for every  $x \in S^{n-1}$ ,  $\|x\|_{\psi_2^n} \leq \sqrt{2/\log n}$ . Indeed, for  $0 \leq x \leq 1$ ,  $n^{x^2/2} \leq nx^2 + 1$ ; hence,

$$\frac{1}{n} \sum_{i=1}^n \exp\left(\frac{x_i^2}{2} \log n\right) = \frac{1}{n} \sum_{i=1}^n n^{x_i^2/2} \leq \frac{1}{n} \sum_{i=1}^n (nx_i^2 + 1) \leq 2.$$

To bound the expectation of  $g$  (with respect to the Haar measure on the sphere), recall the median of the function  $f(x) = \sqrt{n}|x_1|$  satisfies that  $M_f \sim c$ , and that  $\|f\|_{\text{lip}} \leq 1$ . Hence, by concentration of measure on the sphere [MiS], for any  $s > c$  and every  $1 \leq i \leq n$ ,

$$Pr\{x \in S^{n-1} : \sqrt{n}|x_i| \geq 2s\} \leq \sqrt{\frac{\pi}{2}} e^{-s^2/2},$$

and thus,  $\mathbb{E} \exp(cnx_i^2) \leq 2$  for an appropriate absolute constant  $c$ . Recall that there is an absolute constant  $K$  such that for every function  $f$ ,  $\|f\|_{\psi_2} \leq K \mathbb{E} \exp(f^2)$ ; therefore, for  $x = (x_1, \dots, x_n)$ ,

$$\|\sqrt{cn}x\|_{\psi_2^n} \leq \frac{K}{n} \sum_{i=1}^n \exp(cnx_i^2).$$

Taking the expectation with respect to  $x$  on the sphere,

$$\mathbb{E} \|\sqrt{cn}x\|_{\psi_2^n} \leq \frac{K}{n} \sum_{i=1}^n \mathbb{E} \exp(cnx_i^2) \leq K'$$

for an absolute constant  $K'$ .

By the concentration of measure on the sphere applied to the function  $g$ ,

$$Pr\{x \in S^{n-1} : \|x\|_{\psi_2^n} \geq \frac{C}{\sqrt{n}} + t\} \leq \sqrt{\frac{\pi}{2}} e^{-ct^2 n \log n},$$

and the claim follows by selecting  $t = C'/\sqrt{n}$ . ■

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